Sharp bounds for the independence number

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Abstract

For a graph G, the *independence number* $\alpha(G)$ is the size of a largest independent set of G. The maximum degree of the vertices of G is denoted by $\Delta(G)$. We show that for any $1 \le k < n$, any connected graph G on n vertices with $\Delta(G) = k$ has

$$\left\lceil \frac{n-1}{k} \right\rceil \le \alpha(G) \le n - \left\lceil \frac{n-1}{k} \right\rceil,$$

and that these bounds are sharp with the exception that when k divides $n-1$ and G is neither a complete graph nor a cycle, the sharp lower bound is $\frac{n-1}{k} + 1$. From this we immediately obtain sharp bounds for the independence number of any finite graph. Our proof provides an efficient algorithm for constructing an independent set of G whose size is at least the lower bound for $\alpha(G)$.

1   Introduction

Throughout this paper, unless otherwise stated, we shall use capital letters such as X to denote sets or graphs, and small letters such as x to denote elements of a set or positive integers. For any integer $n \ge 1$, $[n]$ denotes the set $\{1, ..., n\}$ of the first n positive integers. For a set X, the set $\{\{x, y\} \colon x, y \in X,\ x \ne y\}$ of all 2-element subsets of X is denoted by $\binom{X}{2}$. In this paper all sets (and graphs) are assumed to be finite.

A *graph G* is a pair $(V(G), E(G))$, where $V(G)$ is a set, called the *vertex set of G*, and $E(G)$ is a subset of $\binom{V(G)}{2}$ and is called the *edge set of G*. If

$\{v, w\}$ is an edge of G, then the vertices v and w are said to be *adjacent* (*in G*), and we say that w is a *neighbour of v* (*in G*) and vice-versa. If any two vertices of G are adjacent (i.e. $E(G) = \binom{V(G)}{2}$), then G is said to be *complete*. An edge $\{v, w\}$ is said to be *incident* to a vertex x if $x = v$ or $x = w$.

A subset I of $V(G)$ is said to be an *independent set of G* if no two vertices in I are adjacent in G. The *independence number of G*, denoted by $\alpha(G)$, is the size of a largest independent set of G.

A subset D of $V(G)$ is said to be a *dominating set of G* if every vertex not in D is adjacent to a vertex in D. Thus, a subset of $V(G)$ is a *maximal independent set of G* (i.e. an independent set that is not a subset of any other one) if and only if it is an independent dominating set. So a largest independent set is a dominating set. For more on domination, see [2].

For any $v \in V(G)$, $N_G(v)$ denotes the set of neighbours of v in G, and $d_G(v)$ denotes the *degree of v in G*, i.e. the size of $N_G(v)$. The maximum vertex degree $\max\{d_G(v) : v \in V(G)\}$ is denoted by $\Delta(G)$.

A graph H is said to be a *sub-graph* of a graph G if $V(H)$ and $E(H)$ are subsets of $V(G)$ and $E(G)$, respectively. For $W \subseteq V(G)$, we denote by $G[W]$ the sub-graph of G induced by W, i.e. $G[W] = \left(W, E(G) \cap \binom{W}{2}\right)$.

If $r \geq 2$ and $v_1, v_2, ..., v_r$ are the distinct vertices of a graph P with $E(P) = \{\{v_i, v_{i+1}\} : i \in [r-1]\}$, then P is called a *$v_1 v_r$-path* or simply a *path*, and we represent P by $\langle v_1, v_2, ..., v_r \rangle$. If $r \geq 3$ and $v_1, v_2, ..., v_r$ are the distinct vertices of a graph C with $E(C) = \{\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{r-1}, v_r\}, \{v_r, v_1\}\}$, then C is called a *cycle*, and we represent C by $(v_1, v_2, ..., v_r)$.

A graph G is said to be *connected* if for any two distinct vertices v and w of G, G has a vw-path as a sub-graph. A *component of G* is a *maximal connected sub-graph of G* (i.e. a connected sub-graph of G that is not a sub-graph of any other one). It is easy to see that if G_1 and G_2 are distinct components of a graph G, then G_1 and G_2 have no common vertices (and hence no common edges).

In this paper we provide a sharp (i.e. attainable) lower bound and a sharp upper bound for the independence number of a connected graph in terms of the maximum vertex degree. From this we immediately obtain sharp bounds for the independence number of any graph. Our proof of the lower bound provides an efficient algorithm for constructing (from the given graph) an independent set whose size is at least the lower bound.

2   A sharp upper bound for $\alpha(G)$

A trivial sharp upper bound for the independence number is given by $\alpha(G) \leq |V(G)| - \delta(G)$, where $\delta(G)$ is the minimum degree of the vertices of G. Indeed,

if v is a vertex in an independent set I of G, then the neighbours of v (of which there are at least $\delta(G)$) are not in I.

We now give a sharp upper bound for the independence number of a connected graph in terms of the number of vertices and the maximum degree $\Delta(G)$, an immediate consequence of which is a sharp upper bound for the independence number of any graph (see Corollary 2.6).

Theorem 2.1 *Let $1 \leq k < n$, where $n = 2$ if $k = 1$. For any connected graph G on n vertices with $\Delta(G) = k$,*

$$\alpha(G) \leq n - \left\lceil \frac{n-1}{k} \right\rceil = \left\lfloor \frac{(k-1)n+1}{k} \right\rfloor.$$

Moreover, the bound is sharp.

Remark 2.2 A connected graph G with $\Delta(G) = 1$ can only consist of two vertices and an edge connecting them. This is why in the above theorem we have the condition that $n = 2$ if $k = 1$.

We start the proof of Theorem 2.1 by making the following observation.

Lemma 2.3 *If I is an independent set of a graph G, then*

$$\sum_{v \in V(G) \setminus I} d_G(v) \geq |E(G)|.$$

Proof. For each $v \in V(G)$, let A_v be the set of those edges of G that are incident to v; so $|A_v| = d_G(v)$. Since I is independent, no edge of G has both vertices in I; in other words, each edge of G has at least one vertex in $V(G) \setminus I$. So $E(G) = \bigcup_{v \in V(G) \setminus I} A_v$. We therefore have

$$|E(G)| = \left| \bigcup_{v \in V(G) \setminus I} A_v \right| \leq \sum_{v \in V(G) \setminus I} |A_v| = \sum_{v \in V(G) \setminus I} d_G(v)$$

as required. $\square$

Corollary 2.4 *If I is an independent set of a connected graph G on n vertices, then*

$$\sum_{v \in V(G) \setminus I} d_G(v) \geq n - 1.$$

Proof. A connected graph G on n vertices has at least $n-1$ edges because if a graph G_2 is obtained by removing an edge from a graph G_1 having exactly c components, then clearly G_2 has at most $c+1$ components, and so we need to remove at least $n-1$ edges from the 1-component graph G to obtain the n-component graph consisting of the vertices of G and no edges. The result now follows by Lemma 2.3. $\qquad\square$

Proof of Theorem 2.1. Let G be a connected graph with $|V(G)| = n$ and $\Delta(G) = k$. Let I be a largest independent set of G. By Lemma 2.4,

$$n - 1 \leq \sum_{v \in V(G)\backslash I} d_G(v) \leq \sum_{v \in V(G)\backslash I} k = |V(G)\backslash I|k = (n - |I|)k.$$

So $|I| \leq n - \left(\frac{n-1}{k}\right)$. Since $|I|$ is an integer and $|I| = \alpha(G)$, we get $\alpha(G) \leq n - \left\lceil \frac{n-1}{k} \right\rceil$ as required.

We now prove that the upper bound is sharp. Consider the following construction.

Construction 2.5 We define a graph $U_{n,k}$ as follows. Let $j = \left\lceil \frac{n-1}{k} \right\rceil$. So $kj = n - 1 + r$ for some integer r such that $0 \leq r \leq k - 1$. Also, $j \geq 1$ since $1 \leq k < n$. Let $m = 1 + j(k-1) - r$; so $m = n - j$. For each $i \in [m]$, let $v_i := i$. For each $l \in [j]$, let $w_l = m + l$. Let $I_{n,k} = \{v_i : i \in [m]\}$ and $J_{n,k} = \{w_l : l \in [j]\}$; so $I_{n,k} = [m]$ and $J_{n,k} = [n]\backslash I_{n,k}$. If $j \geq 2$, then let $F_p = \{\{v_{1+(p-1)(k-1)+i}, w_p\} : 0 \leq i \leq k - 1\}$ for each $p \in [j-1]$. Let $F_j = \{\{v_{1+(j-1)(k-1)+i}, w_j\} : 0 \leq i \leq k - 1 - r\}$. Take $U_{n,k}$ to be the (bipartite) graph with $V(G) = I_{n,k} \cup J_{n,k} = [n]$ and $E(U_{n,k}) = F_1 \cup ... \cup F_j$.

If a graph S has a vertex v^* such that $E(S) = \{\{v^*, v\} : v \in V(S)\backslash\{v^*\}\}$, then S is called a *star* with *center* v^*. Thus, for any $p \in [j]$, the edges in F_p form a star sub-graph S_p of $U_{n,k}$ with center w_p, and if $p \leq j - 1$ then $V(S_p) \cap V(S_{p+1}) = \{v_{1+p(k-1)}\}$ (since $\{v_{1+p(k-1)}, w_p\} \in F_p$ and $\{v_{1+p(k-1)}, w_{p+1}\} \in F_{p+1}$). So $U_{n,k}$ is a connected graph because stars are connected graphs, every vertex of $U_{n,k}$ is a vertex of at least one of $S_1, ..., S_j$, and S_p is connected to S_{p+1} via $v_{1+p(k-1)}$ for any $p \in [j]\backslash\{j\}$. Thus, since $\Delta(U_{n,k}) = d_G(w_1) = k$ (note that if $j = 1$, then $n = k + 1$ and $r = 0$), the upper bound we proved above gives us $\alpha(U_{n,k}) \leq n - \left\lceil \frac{n-1}{k} \right\rceil$. Now, by construction, $I_{n,k}$ is an independent set of $U_{n,k}$ of size $n - \left\lceil \frac{n-1}{k} \right\rceil$. Therefore, $\alpha(U_{n,k}) = \left\lceil \frac{n-1}{k} \right\rceil$. $\qquad\square$

A graph that consists of only one vertex is called a *singleton*. For any graph G, we denote the set of non-singleton components of G by $\mathcal{C}(G)$.

Corollary 2.6 *If G is a graph on n vertices, then*

$$\alpha(G) \leq n - \sum_{H \in \mathcal{C}(G)} \left\lceil \frac{|V(H)| - 1}{\Delta(H)} \right\rceil,$$

and equality holds if for any $H \in \mathcal{C}(G)$, H is a copy of $U_{|V(H)|, \Delta(H)}$.

Proof. Let s be the number of singleton components of a graph G. If $\mathcal{C}(G) = \emptyset$ then $\alpha(G) = s = n$. Suppose $\mathcal{C}(G) \neq \emptyset$. By Theorem 2.1, for any $H \in \mathcal{C}(G)$ we have $\alpha(H) \leq |V(H)| - \left\lceil \frac{|V(H)|-1}{\Delta(H)} \right\rceil$. Now clearly

$$\alpha(G) = s + \sum_{H \in \mathcal{C}(G)} \alpha(H) \leq s + \sum_{H \in \mathcal{C}(G)} \left(|V(H)| - \left\lceil \frac{|V(H)| - 1}{\Delta(H)} \right\rceil \right)$$

$$= s + \sum_{H \in \mathcal{C}(G)} |V(H)| - \sum_{H \in \mathcal{C}(G)} \left\lceil \frac{|V(H)| - 1}{\Delta(H)} \right\rceil = n - \sum_{H \in \mathcal{C}(G)} \left\lceil \frac{|V(H)| - 1}{\Delta(H)} \right\rceil,$$

and by Theorem 2.1, equality holds throughout if each $H \in \mathcal{C}(G)$ is a copy of $U_{|V(H)|, \Delta(H)}$. $\square$

We mention that there other upper bounds in the literature; for example, in [3] an upper bound for $\alpha(G)$ in terms of the eigenvalues of the *Laplacian matrix* of G is given.

3   A sharp lower bound for $\alpha(G)$

Caro [1] and Wei [4] independently obtained the following classical lower bound for $\alpha(G)$:

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d_G(v) + 1}.$$

So $\alpha(G) \geq \frac{|V(G)|}{\Delta(G)+1}$ and hence, since $\alpha(G)$ is an integer, $\alpha(G) \geq \left\lceil \frac{|V(G)|}{\Delta(G)+1} \right\rceil$; but this is also an immediate consequence of the fact that any vertex in a maximal independent set I of G has at most $\Delta(G)$ neighbours (not in I) and that, since I is a dominating set (see Section 1), $V(G)$ consists of the vertices in I together with their neighbours. This bound is attained by a graph G with c components, each of which is a complete graph on $k + 1$ vertices; clearly such a graph has $|V(G)| = c(k + 1) = \alpha(G)(\Delta(G) + 1)$.

We now show that for a connected graph G, the lower bound $\alpha(G) \geq \left\lceil \frac{|V(G)|}{\Delta(G)+1} \right\rceil$ improves to $\alpha(G) \geq \left\lceil \frac{|V(G)|-1}{\Delta(G)} \right\rceil$, and if G is neither a complete

graph nor a cycle, then $\alpha(G) \geq \left\lceil \frac{|V(G)|}{\Delta(G)} \right\rceil$; these new bounds are sharp. It is worth pointing out that a maximal independent set of a connected graph G may be of size less than $\left\lceil \frac{|V(G)|-1}{\Delta(G)} \right\rceil$ (only a largest one is guaranteed to have at least $\left\lceil \frac{|V(G)|-1}{\Delta(G)} \right\rceil$ elements).

Example 3.1 If $m \geq 1$ and G is the graph with $V(G) = [4m+2]$ and $E(G) = \{\{i, (i+j) \bmod n\} \colon i \in [n], j \in [m]\}$, then G is connected, $\Delta(G) = 2m$, and $\{1, 2m+2\}$ is a maximal independent set of G of size $2 < \left\lceil \frac{|V(G)|-1}{\Delta(G)} \right\rceil$.

The proof of our result provides an efficient algorithm for constructing an independent set of size at least $\left\lceil \frac{|V(G)|-1}{\Delta(G)} \right\rceil$ from a connected graph G. It also yields an efficient (but longer) algorithm for constructing an independent set of size at least $\left\lceil \frac{|V(G)|}{\Delta(G)} \right\rceil$ for the case when G is neither a complete graph nor a cycle (see Section 4). The case when $\Delta(G)$ divides $|V(G)| - 1$ is significantly more challenging than the complementary case.

Theorem 3.2 *Let $1 \leq k < n$, where $n = 2$ if $k = 1$.*
(a) For any connected graph G on n vertices which has $\Delta(G) = k$,

$$\alpha(G) \geq \left\lceil \frac{n-1}{k} \right\rceil. \tag{1}$$

Moreover, if k does not divide $n - 1$, then the bound is sharp.
(b) If k divides $n - 1$, then for any connected graph G on n vertices which has $\Delta(G) = k$ and is neither a complete graph nor a cycle,

$$\alpha(G) \geq \frac{n-1}{k} + 1. \tag{2}$$

Moreover, the bound is sharp.

Proof. We will first prove (1), then (2), and finally the sharpness in both (a) and (b).

Step 1: Proof of (1). Since $\Delta(G) = k$, there is a vertex v_1 of G with $|N_G(v_1)| = k$. Let $I_1 = \{v_1\}$ and $V_1 = \{v_1\} \cup N_G(v_1)$. So $|V_1| = 1 + |N_G(v_1)|$. If $V(G) = V_1$, then $|I_1| = 1 = \frac{|N_G(v_1)|}{|N_G(v_1)|} = \frac{n-1}{k}$.

Suppose $V(G) \neq V_1$. Then, since G is connected, there is a vertex v_2 in $V(G) \backslash V_1$ that is adjacent to some vertex p_1 in V_1 (otherwise there is no path from a vertex in V_1 to a vertex in $V(G) \backslash V_1$). Since $v_2 \notin V_1$ and $N_G(v_1) \subseteq V_1$, v_2 is not adjacent to v_1. Let $I_2 = \{v_1, v_2\}$, $V_2 = V_1 \cup \{v_2\} \cup N_G(v_2)$ and

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$W_2 = V_2 \backslash V_1$. So $V_2 = I_2 \cup N_G(v_1) \cup N_G(v_2)$ and $V_2 = V_1 \cup W_2$. Since p_1 is in both V_1 and $N_G(v_2)$, we have $|W_2| \leq |\{v_2\} \cup N_G(v_2)| - 1 = |N_G(v_2)|$. If $V(G) = V_2$, then $n = |V_1 \cup W_2| \leq 1 + |N_G(v_1)| + |N_G(v_2)| \leq 1 + 2k$, $|I_2| = 2 = \frac{2k}{k} \geq \frac{n-1}{k}$ and hence, since $|I_2|$ is an integer, $|I_2| \geq \lceil \frac{n-1}{k} \rceil$. Since I_2 is an independent set of G, $\alpha(G) \geq \lceil \frac{n-1}{k} \rceil$.

Suppose $V(G) \neq V_2$. Then, since G is connected, there is a vertex v_3 in $V(G) \backslash V_2$ that is adjacent to some vertex p_2 in V_2. Since $v_3 \notin V_2$ and $N_G(v_1) \cup N_G(v_2) \subseteq V_2$, v_3 is not adjacent to any of v_1 and v_2. Let $I_3 = \{v_1, v_2, v_3\}$, $V_3 = V_2 \cup \{v_3\} \cup N_G(v_3)$ and $W_3 = V_3 \backslash V_2$. So $V_3 = I_3 \cup N_G(v_1) \cup N_G(v_2) \cup N_G(v_3)$ and $V_3 = V_2 \cup W_3 = V_1 \cup W_2 \cup W_3$. Since p_2 is in both V_2 and $N_G(v_3)$, we have $|W_3| \leq |\{v_3\} \cup N_G(v_3)| - 1 = |N_G(v_3)|$. Repetition of this procedure stops when we obtain an independent set I_m of m distinct vertices $v_1, v_2, ..., v_m$ such that the set $V_m = I_m \cup N_G(v_1) \cup ... \cup N_G(v_m) = V_1 \cup W_1 \cup ... \cup W_m$ is the whole vertex set $V(G)$, and this yields

$$n = |V_m| = \left| V_1 \cup \bigcup_{i=2}^{m} W_i \right| \leq (1 + k) + \sum_{i=2}^{m} |N_G(v_i)| \leq 1 + mk.$$

Therefore,

$$|I_m| = m \geq \frac{n-1}{k} \tag{3}$$

and hence (1).

Step 2: Proof of (2). We will use induction to prove (1) and (2) combined for graphs that are neither complete graphs nor cycles. So we prove that if a connected graph G on n vertices has $\Delta(G) = k$ and is neither a complete graph nor a cycle, then $\alpha(G) \geq \lceil \frac{n}{k} \rceil$. Thus, let G be a connected graph that is neither a complete graph nor a cycle. By Remark 2.2, $k \geq 2$.

Consider first $k = 2$. Clearly a connected graph with maximum vertex degree 2 can only be a path or a cycle. So G is a path $\langle a_1, ..., a_n \rangle$ and clearly $\{a_{2(i-1)+1} : i \in \{1, 2, ..., \lceil \frac{n}{2} \rceil\}\}$ is an independent set of G of maximum size; so $\alpha(G) = \lceil \frac{n}{2} \rceil$.

We now consider $k \geq 3$ and prove the result by induction on n. Consider the base case $n = k + 1$. Since G is not complete, $d_G(v) < k$ for some $v \in V(G)$, and hence G has a vertex w that is not adjacent to v. So $\{v, w\}$ is an independent set of G of size $2 = \lceil \frac{n}{k} \rceil$, and hence $\alpha(G) \geq \lceil \frac{n}{k} \rceil$.

Now consider $n \geq k + 2$. We apply the procedure in Step 1, which yields the independent set $I_m = \{v_1, ..., v_m\}$. Since $n \geq k + 2$, $m \geq 2$ by (3). Since $v_m \in W_m$ and $|W_m| \leq |N_G(v_m)|$, $1 \leq |W_m| \leq k$. Let $H = G[V_{m-1}]$ and $n' = |V(H)| = |V_{m-1}|$. Then $n' = n - |W_m|$ (since $V_m = V(G)$) and H is a connected graph (by the procedure). Since $d_H(v_1) = d_G(v_1) = k$, $\Delta(H) = k$.

Since $k \geq 3$ and each vertex of a cycle has degree 2, H is not a cycle. Suppose H is complete. Then $n' = k + 1$ as $\Delta(H) = k$. Since $\Delta(G) = k$, we also get that the neighbours in G of any vertex in V_{m-1} are the k other vertices in V_{m-1}, and hence no vertex in W_m is adjacent to a vertex in V_{m-1}; but then G is not connected, a contradiction. So H is not complete. We can therefore apply the inductive hypothesis to obtain $\alpha(H) \geq \lceil \frac{n'}{k} \rceil$. So H has an independent set J of size at least $\lceil \frac{n'}{k} \rceil$.

If k does not divide $n - 1$, then $\lceil \frac{n}{k} \rceil = \lceil \frac{n-1}{k} \rceil$ and hence, by (1), $\alpha(G) \geq \lceil \frac{n}{k} \rceil$. So suppose k divides $n - 1$.

It is clear from Step 1 that the inequality $|I_m| \geq \frac{n-1}{k}$ in (3) is strict if and only if $|W_i| < k$ for some $i \in \{2, ..., m\}$. Thus, if $|W_i| < k$ for some $i \in \{2, ..., m\}$, then $|I_m| \geq \frac{n-1}{k} + 1$ (since $|I_m|$ is an integer and k divides $n - 1$) and hence $\alpha(G) \geq \lceil \frac{n}{k} \rceil$.

Now suppose $|W_i| = k$ for any $i \in \{2, ..., m\}$; so $m = \frac{n-1}{k}$ and $n' = n - k$. Suppose G has two non-adjacent vertices x and x' in W_m. Since each neighbour of each of the vertices in I_{m-1} is in V_{m-1}, no vertex in W_m is adjacent to a vertex in I_{m-1}. Thus, $I_{m-1} \cup \{x, x'\}$ is an independent set of G of size $m + 1 = \frac{n-1}{k} + 1 = \lceil \frac{n}{k} \rceil$ and hence $\alpha(G) \geq \lceil \frac{n}{k} \rceil$.

Finally, suppose any two distinct vertices in W_m are adjacent in G. Then, since $\Delta(G) = k$, each vertex in W_m has at most one neighbour in V_{m-1} as it has $k - 1$ neighbours in W_m. Since G is connected and $V_{m-1} = V(G) \backslash W_m$, G has a vertex w in W_m that has a (unique) neighbour y in V_{m-1} (in fact, v_m does). Since y is adjacent to one of $v_1, ..., v_{m-1}$ (by definition of V_{m-1}) and has at most $k - 1$ other neighbours, and since $|W_m| = k$, there exists a vertex w' in W_m that is not adjacent to y.

Suppose w' has no neighbour in V_{m-1}. Then $J \cup \{w'\}$ is an independent set of G of size at least $\lceil \frac{n'}{k} \rceil + 1 = \lceil \frac{n-k}{k} \rceil + 1 = \lceil \frac{n-1}{k} - \frac{k-1}{k} \rceil + 1$. Since k divides $n - 1$, $\lceil \frac{n-1}{k} - \frac{k-1}{k} \rceil = \frac{n-1}{k}$ and hence $\alpha(G) \geq \frac{n-1}{k} + 1 = \lceil \frac{n}{k} \rceil$.

Now suppose w' has a neighbour y' in V_{m-1}. By choice of w', $y' \neq y$. Let H' be the graph obtained by adding $\{y, y'\}$ to the set of edges of H (so $H' = H$ if $\{y, y'\}$ is already in $E(H)$). H' is connected as H is connected. Since each of y and y' has at most k neigbours in G and at least one neighbour in W_m, we have $d_{H'}(y) \leq k$ and $d_{H'}(y') \leq k$; so $\Delta(H') \leq \Delta(G) = k$. But $\Delta(H') \geq \Delta(H) = k$, so $\Delta(H') = k$. Since $k \geq 3$, H' is not a cycle.

Suppose H' is not complete. Then, by the inductive hypothesis, $\alpha(H') \geq \lceil \frac{n'}{k} \rceil$ and hence H' has an independent set J' of size at least $\lceil \frac{n'}{k} \rceil$. If $y \in J'$ then $y' \notin J'$ (since $\{y, y'\} \in E(H')$) and hence, since y' is the unique neighbour of w' in V_{m-1}, $J' \cup \{w'\}$ is an independent set of G, from which we again obtain $\alpha(G) \geq \lceil \frac{n'}{k} \rceil + 1 = \lceil \frac{n}{k} \rceil$. If $y \notin J'$ then, since y is the unique neighbour of w in V_{m-1}, $J' \cup \{w\}$ is an independent set of G and

hence $\alpha(G) \geq \left\lceil \frac{n}{k} \right\rceil$.

Now suppose H' is complete. Then $n' = k + 1$ since $\Delta(H') = k$. Since $\Delta(G) = k$, we also deduce that $N_G(v) = V_{m-1} \backslash \{v\}$ for any $v \in V_{m-1} \backslash \{y, y'\}$, $N_G(y) = (V_{m-1} \backslash \{y, y'\}) \cup \{w\}$ and $N_G(y') = (V_{m-1} \backslash \{y, y'\}) \cup \{w'\}$. Thus, since $|W_m| = k \geq 3$, there exists a vertex w'' in W_m that is not a neighbour of any vertex in V_{m-1}. So $\{y, y', w''\}$ is an independent set of G of size 3. Now $\left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{n'+k}{k} \right\rceil = \left\lceil \frac{2k+1}{k} \right\rceil = 3$. So $\alpha(G) \geq \left\lceil \frac{n}{k} \right\rceil$.

Step 3: Proof of the fact that the bound in (b) is sharp and that the bound in (a) is sharp when k does not divide $n - 1$. Consider the following construction.

Construction 3.3 We define a graph $L_{n,k}$ as follows. Let $j = \left\lceil \frac{n}{k} \right\rceil$. Let $X_i = \{s \in [n] \colon (i-1)k + 1 \leq s \leq ik\}$ for each $i \in [j-1]$, and let $X_j = \{s \in [n] \colon (j-1)k + 1 \leq s \leq n\}$. For each $i \in [j-1]$, let $e_i = \{ik, ik+1\}$ (so e_i is an edge with one vertex in X_i and the other in X_{i+1}). Now take $V(L_{n,k}) = [n]$ and $E(L_{n,k}) = \binom{X_1}{2} \cup \binom{X_2}{2} \cup ... \cup \binom{X_j}{2} \cup \{e_1, ..., e_{j-1}\}$.

So $L_{n,k}$ consists of $j = \left\lceil \frac{n}{k} \right\rceil$ complete sub-graphs induced by the disjoint sets $X_1, ..., X_j$, respectively, together with the $j - 1$ edges $e_1, ..., e_{j-1}$, where e_i connects the complete sub-graph on X_i to the one on X_{i+1}.

We are not concerned with the trivial case $k = 1$ since it gives us that k divides $n - 1$ and G must be the complete graph on 2 vertices. So consider $k \geq 2$. Then $L_{n,k}$ is neither a complete graph nor a cycle. The edges incident to vertex k are $\{1, k\}, ..., \{k-1, k\}$ and e_1, and clearly $\Delta(L_{n,k}) = d_{L_{n,k}}(k) = k$. Let $I_{n,k} = \{i \in [j] \colon (i-1)k + 1\}$. For any $i \in [j]$, any independent set of $L_{n,k}$ can have at most one vertex in X_i; so $I_{n,k}$ is an independent set of $L_{n,k}$ of maximum size and hence $\alpha(L_{n,k}) = \left\lceil \frac{n}{k} \right\rceil$. If k does not divide $n - 1$, then $\left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{n-1}{k} \right\rceil$ and hence $\alpha(L_{n,k}) = \left\lceil \frac{n-1}{k} \right\rceil$. If k divides $n - 1$, then $\left\lceil \frac{n}{k} \right\rceil = \frac{n-1}{k} + 1$ and hence $\alpha(L_{n,k}) = \frac{n-1}{k} + 1$. $\qquad\square$

Remark 3.4 If G is a complete graph, then $n = k + 1$ and any vertex of G forms an independent set of G of size $\alpha(G) = 1 = \frac{n-1}{k} = \left\lceil \frac{n-1}{k} \right\rceil$. If G is a cycle $(v_1, v_2, ..., v_n)$, then clearly $\{v_{2(i-1)+1} \colon i \in \{1, ..., \left\lceil \frac{n-1}{2} \right\rceil\}\}$ is an independent set of G of maximum size and hence $\alpha(G) = \left\lceil \frac{n-1}{2} \right\rceil$. Thus, by Theorem 3.2, the only connected graphs which attain the bound in (1) with k a divisor of $n - 1$ are the cycles ($k = 2$) on an odd number of vertices and the complete graphs.

Theorem 3.2 has the following immediate consequence, which has actually been proved in the proof of Theorem 3.2 itself.

Corollary 3.5 *If a connected graph G is neither a complete graph nor a cycle, then*

$$\alpha(G) \geq \left\lceil \frac{|V(G)|}{\Delta(G)} \right\rceil,$$

and equality holds if $G = L_{n,k}$.

As in Section 2, we denote the set of non-singleton components of a graph G by $\mathcal{C}(G)$ and we conclude this section with a lower bound for any graph.

Corollary 3.6 *If G is a graph on n vertices, then*

$$\alpha(G) \geq n - \sum_{H \in \mathcal{C}(G)} \left\lfloor \frac{(\Delta(H) - 1)|V(H)| + 1}{\Delta(H)} \right\rfloor.$$

Proof. Let s be the number of singleton components of a graph G. Clearly

$$\alpha(G) = s + \sum_{H \in \mathcal{C}(G)} \alpha(H) = n - \sum_{H \in \mathcal{C}(G)} |V(H)| + \sum_{H \in \mathcal{C}(G)} \alpha(H)$$

$$\geq n - \sum_{H \in \mathcal{C}(G)} |V(H)| + \sum_{H \in \mathcal{C}(G)} \left\lceil \frac{|V(H)| - 1}{\Delta(H)} \right\rceil \quad \text{(by Theorem 2.1)}$$

$$= n - \sum_{H \in \mathcal{C}(G)} \left(|V(H)| - \left\lceil \frac{|V(H)| - 1}{\Delta(H)} \right\rceil \right)$$

and hence the result. $\qquad\square$

4 Constructing an independent set of size at least $\left\lceil \frac{|V(G)|}{\Delta(G)} \right\rceil$

In the following, we will refer to Step 1 and Step 2 of the proof of Theorem 3.2 simply by Step 1 and Step 2, respectively.

Let G be a connected graph on n vertices with $\Delta(G) = k$. As explained in Remark 3.4, if G is complete or G is a cycle, then constructing a largest independent set is trivial. Suppose G is neither a complete graph nor a cycle. Then, by Corollary 3.5, $\alpha(G) \geq \left\lceil \frac{n}{k} \right\rceil$ and the bound is sharp. For the case when k does not divide $n-1$, Step 1 provides a quick algorithm for obtaining an independent set of size at least $\left\lceil \frac{n-1}{k} \right\rceil = \left\lceil \frac{n}{k} \right\rceil$. For the remaining case when k divides $n - 1$, the inductive argument used in Step 2 indicates an efficient algorithm for constructing an independent set of size at least $\left\lceil \frac{n}{k} \right\rceil$.

Indeed, suppose k divides $n - 1$. We first apply the procedure in Step 1. Let $I_0 = \emptyset$ and $W_1 = V_1$.

If $m = 1$ then $n = k + 1$, and the algorithm for this trivial case is given in Step 2.

Now consider $m \geq 2$. As explained in Step 2, if $|W_i| < k$ for some $i \in \{2, ..., m\}$, then $|I_m| \geq \lceil \frac{n}{k} \rceil$.

Now suppose $|W_i| = k$ for any $i \in \{2, ..., m\}$; so $m = \frac{n-1}{k}$. As we showed in Step 2, if G has two non-adjacent vertices x and x' in W_m, then $I_{m-1} \cup \{x, x'\}$ is an independent set of G of size $\frac{n-1}{k} + 1 = \lceil \frac{n}{k} \rceil$.

Finally, suppose any two distinct vertices in W_m are adjacent in G. Let $G_m = G$ and $G'_m = G_m$. Let H, w, w' and y be as in Step 2, and let us label these G_{m-1}, w_m, w'_m and y_{m-1}, respectively. Note that we can write $G_{m-1} = G'_m[V_{m-1}]$. As explained in Step 2, if w'_m has a neighbour $y' = y'_{m-1}$ in V_{m-1}, then $y'_{m-1} \neq y_{m-1}$ and we define G'_{m-1} by $V(G'_{m-1}) = V(G_{m-1})$ and $E(G'_{m-1}) = E(G_{m-1}) \cup \{\{y_{m-1}, y'_{m-1}\}\}$. If w'_m has no neighbour in V_{m-1}, then we take $y'_{m-1} = y_{m-1}$ and $G'_{m-1} = G_{m-1}$. If $W_{m-1} \neq W_1$ and any two distinct vertices in W_{m-1} are adjacent in G'_{m-1}, then we obtain $w_{m-1}, w'_{m-1}, y_{m-2}, y'_{m-2}$, $G_{m-2} = G'_{m-1}[V_{m-2}]$ and G'_{m-2} from G'_{m-1} similarly to the way we obtained $w_m, w'_m, y_{m-1}, y'_{m-1}$, G_{m-1} and G'_{m-1} from G'_m, and we keep on repeating this until we obtain a graph G'_{r-1} such that either
(i) G'_{r-1} has two non-adjacent vertices x and x' in W_{r-1} or
(ii) $r = 2$.

Suppose (i) holds. Then, setting $I'_{r-1} = I_{r-2} \cup \{x, x'\}$, we have that I'_{r-1} is an independent set of G'_{r-1} of size r. The main fact that we need to recall is that for any $r \leq i \leq m$, y_{i-1} is the unique neighbour of w_i in V_{i-1}, and if y'_{i-1} is a neighbour of w'_i, then w'_i has no other neighbours in V_{i-1}, $y'_{i-1} \neq y_{i-1}$ and y'_{i-1} is adjacent to y_{i-1} in G'_{i-1}. If $y_{r-1} \neq y'_{r-1}$, then $\{y_{r-1}, y'_{r-1}\} \in E(G'_{r-1})$, at least one of y_{r-1} and y'_{r-1} is not in $\{x, x'\}$, and we take $I'_r = I'_{r-1} \cup \{w''_r\}$, where $w''_r = w_r$ if $y_{r-1} \notin \{x, x'\}$, and $w''_r = w'_r$ if $y'_{r-1} \notin \{x, x'\}$. If $y_{r-1} = y'_{r-1}$, then we take $I'_r = I'_{r-1} \cup \{w'_r\}$. So I'_r is an independent set of G'_r (and also of G_r) of size $r + 1$. If $m = r$, then we are done. Suppose $m \geq r + 1$. If $y_r \in I'_r$, then we take $I'_{r+1} = I'_r \cup \{w'_{r+1}\}$, otherwise we take $I'_{r+1} = I'_r \cup \{w_{r+1}\}$. So I'_{r+1} is an independent set of G'_{r+1} (and also of G_{r+1}) of size $r + 2$. We go on like this until we obtain an independent set I'_m of G of size $m + 1$.

Now suppose (i) does not hold. Then $r = 2$ and any two vertices in $G'_{r-1} = G'_1$ are adjacent (that is, G'_1 is complete). As shown in the last part of Step 2, there exists a vertex w''_2 in W_2 that has no neighbour in V_1 and $\{y_1, y'_1, w''_2\}$ is an independent set of G_2 of size 3. If $m = 2$, then we are done. Suppose $m \geq 3$. Let $I'_2 = \{y_1, y'_1, w''_2\}$. From the last part of Step 2, we know that the neighbours of y_1 and y'_1 are all in V_2. Thus, since y_2 and y'_2 have neighbours in W_3, we have $y_2, y'_2 \notin \{y_1, y'_1\}$. So at most one of y_2 and y'_2 is in

I_2'. If $y_2 \in I_2'$, then we take $I_3' = I_2' \cup \{w_3'\}$, otherwise we take $I_3' = I_2' \cup \{w_3\}$. So I_3' is an independent set of G_3' (and also of G_3) of size 4. We can now extend this to an independent set I_m' of G of size $m + 1$ as above.

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